

Conservation laws for collisions of branes (or shells) in general relativity

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We consider the collision of self-gravitating n -branes in a $(n+2)$ -dimensional spacetime. We show that there is a geometrical constraint which can be expressed as a simple sum rule for angles characterizing Lorentz boosts between branes and the intervening spacetime regions. This constraint can then be re-interpreted as either energy or momentum conservation at the collision.

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I. INTRODUCTION

Conservation laws during collisions are a cornerstone of Newtonian mechanics. They have been generalized to special relativity and are ubiquitous in the interpretation of collider experiments. In general relativity, the question of conservation laws during collisions is subtler because the colliding (self-gravitating) objects affect spacetime itself. Some attention has been paid in the literature to the collision of shells in general relativity [1–8]. Only particular cases however have been considered and the calculations are rather involved. One can distinguish two types of approaches, one based on (not always well justified) energy-momentum conservation laws, another based on geometrical constraints, especially in the case of light-like shells. The interest for collisions in general relativity has been renewed very recently in the context of brane-world cosmology by the idea that our current universe could be the product of a collision of 3-branes in a five-dimensional spacetime [9–11].

In this Letter, we propose a unified treatment, based on a purely geometric approach, which considerably simplifies the calculations, and thus immediately applies to *any number* of n -branes in a $D = n+2$ dimensional spacetime. The case $n = 2$ corresponds to shells in standard general relativity, whereas the case $n = 3$ applies to the collision of 3-branes in the brane cosmology scenarios. The simplicity of our treatment follows from expressing the geometrical constraint as a sum rule for angles associated with generalised Lorentz boosts between the branes and the intervening spacetime regions.

II. LOCAL MOTION OF A BRANE

In an $n+2$ dimensional spacetime, n -branes divide the spacetime into distinct regions. We will consider the

simplest case where each such region is empty and can be described by a metric of the form

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2 d\Omega_n^2, \quad (1)$$

where the ‘orthogonal’ metric $d\Omega_n^2$ does not depend on either T or R . The well-known case of a Schwarzschild-(anti)-de Sitter spacetime corresponds to $f(R) = k - (\mu/R^{n-1}) \mp (R/\ell)^2$.

A brane at the boundary of this region is described by a two-dimensional trajectory $(T(\tau), R(\tau))$, where τ is the proper time. If we define the two-dimensional velocity vector $u^a = (\dot{T}, \dot{R})$, where the dot denotes the derivative with respect to τ , then by definition of the proper time, u^a is normalized so that $g_{ab}u^a u^b = -\dot{T}^2 + \dot{R}^2 = -1$. One can make connection with the formulas of special relativity by introducing a basis of normalized vectors, $\mathbf{e}_T = f^{-1/2} \frac{\partial}{\partial T}$ and $\mathbf{e}_R = \sqrt{f} \frac{\partial}{\partial R}$.

One can then define a Lorentz factor $\gamma = -\mathbf{e}_T \cdot \mathbf{u}$ and a relative velocity β , given by $\gamma\beta = \mathbf{e}_R \cdot \mathbf{u}$, which yields

$$\gamma = \sqrt{1 + \frac{\dot{R}^2}{f}}, \quad \gamma\beta = \frac{\epsilon \dot{R}}{\sqrt{f}}. \quad (2)$$

where $\epsilon = +1$ if R decreases from ‘left’ to ‘right’, $\epsilon = -1$ otherwise. Equation (2) characterizes the motion of the brane \mathcal{B} with respect to an observer at rest in the frame \mathcal{R} defined by (1). It is easy to check that this implies the standard special relativistic formula $\gamma = 1/\sqrt{1 - \beta^2}$.

At any point along the brane trajectory there is a local transformation from the bulk coordinates T and R to the proper time along the brane, τ , and Gaussian normal coordinate, χ :

$$\begin{pmatrix} d\tau \\ d\chi \end{pmatrix} = \Lambda(-\alpha) \begin{pmatrix} \sqrt{f}dT \\ \epsilon dR/\sqrt{f} \end{pmatrix}, \quad (3)$$

where $\Lambda(\theta)$ is a two-dimensional Lorentz matrix

$$\Lambda(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}, \quad (4)$$

and α in Eq. (3) is the Lorentz angle associated with the motion of the brane with respect to the original coordinate systems \mathcal{R} , i.e.

$$\alpha = \sinh^{-1}(\epsilon \dot{R}/\sqrt{f}). \quad (5)$$

III. JUNCTION CONDITIONS

Being of codimension 1, the worldsheet of each brane we consider here will separate the spacetime in two disconnected regions: the left region, which we call \mathcal{R}_- , and the right region, which we call \mathcal{R}_+ , with two metrics of the form (1) on the two sides. The coordinate R must be the same on the two sides because the orthogonal part of the metric must be continuous.

The junction conditions can be written in the form [12]

$$[K_{AB}] = -\kappa^2 \left(S_{AB} - \frac{S}{n} g_{AB} \right), \quad (6)$$

where the left hand side is the jump of the extrinsic curvature tensor across the brane. S_{AB} is the energy-momentum tensor of the brane, S its trace, and κ^2 is the coupling between matter and gravity. For the orthogonal part, the extrinsic curvature components are $K_{ij} = (\epsilon/R)\sqrt{f + \dot{R}^2}g_{ij}$, which in the junction conditions yields the expression

$$\epsilon_+ \sqrt{f_+ + \dot{R}^2} - \epsilon_- \sqrt{f_- + \dot{R}^2} = \frac{\kappa^2}{n} \rho R, \quad (7)$$

where ρ is the comoving energy density on the brane. This can be translated into a Friedmann-like equation inside the brane, which reads

$$\dot{R}^2 = \frac{\kappa^4}{4n^2} \rho^2 R^2 - \frac{1}{2} (f_+ + f_-) + \frac{n^2}{\kappa^4 \rho^2 R^2} (f_+ - f_-)^2. \quad (8)$$

The other part of the junction conditions is equivalent to the usual energy conservation law $\dot{\rho} + n(\dot{R}/R)(\rho + P) = 0$, where P is the pressure.

Let us now study the coordinate transformation that relates the coordinates (T_+, R_+) to the coordinates (T_-, R_-) (note that, for the brane, $R_- = R_+ = R$, as imposed by the continuity of the metric along the ‘orthogonal’ directions). Because for both coordinate systems the metric is of the diagonal form (1), the coordinate transformation is necessarily given by

$$\begin{pmatrix} \sqrt{f_+} dT_+ \\ \epsilon_+ dR_+ / \sqrt{f_+} \end{pmatrix} = \Lambda(\alpha) \begin{pmatrix} \sqrt{f_-} dT_- \\ \epsilon_- dR_- / \sqrt{f_-} \end{pmatrix}, \quad (9)$$

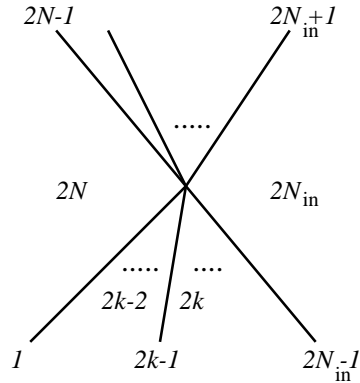


FIG. 1. Collision of N_{in} ingoing branes, yielding $N_{out} = N - N_{in}$ outgoing branes. Odd integers denote branes and even integers denote regions in between.

where Λ is a two-dimensional Lorentz matrix as defined in Eq. (4). If one evaluates the coordinate transformation *at the brane*, it is easy to see that the angle is given by $\alpha = \alpha_+ - \alpha_-$, where α_+ and α_- are the Lorentz angles associated with the motion of the brane with respect to the coordinate systems \mathcal{R}_+ and \mathcal{R}_- respectively, as defined in Eq. (5). Intuitively, this result is very easy to understand. It simply means that to go from the coordinates of the region \mathcal{R}_- to the coordinates of the region \mathcal{R}_+ , one must do a (pseudo-)Lorentz transformation, which is the combination of a Lorentz transformation going from \mathcal{R}_- to a system where the brane is at rest, with a Lorentz transformation from the brane system to \mathcal{R}_+ .

IV. SYSTEM OF SEVERAL BRANES

So far, we have considered only one brane and the two regions surrounding it. To describe the collision of a system of branes in general, we now introduce a system of $N = N_{in} + N_{out}$ branes, consisting of N_{in} ingoing branes colliding simultaneously and of N_{out} outgoing branes, which are produced by the collision. These N branes are separated by N different regions of spacetime, which are assumed to be empty but can be endowed with different cosmological constants and Schwarzschild masses.

To simplify the formalism, we are going to label alternately branes and regions by integers, starting from the leftmost ingoing brane and going anticlockwise around the point of collision (see Fig. 1). The branes will thus be denoted by odd integers, $2k-1$ ($1 \leq k \leq N$), and the regions by even integers, $2k$ ($1 \leq k \leq N$). Let us introduce, as before, the angle $\alpha_{2k-1|2k}$ which characterizes the motion of the brane \mathcal{B}_{2k-1} with respect to the region \mathcal{R}_{2k} , and which is defined by

$$\sinh \alpha_{2k-1|2k} = \frac{\epsilon_{2k} \dot{R}_{2k-1}}{\sqrt{f_{2k}}}. \quad (10)$$

Of course we can equally describe the motion of the region \mathcal{R}_{2k} with respect to the brane by the Lorentz angle

$$\alpha_{2k|2k-1} = -\alpha_{2k-1|2k}.$$

We will find it convenient to define a rescaled brane density

$$\tilde{\rho}_{2k-1} = \pm \frac{\kappa^2}{n} \rho_{2k-1} R, \quad (11)$$

with the plus sign for ingoing branes ($1 \leq k \leq N_{in}$), the minus sign for outgoing branes ($N_{in} + 1 \leq k \leq N$). An outgoing positive energy density brane thus has the same sign as an ingoing negative energy density brane.

The junction condition (7) then takes the simple form,

$$\begin{aligned} \tilde{\rho}_{2k-1} &= \epsilon_{2k} \sqrt{f_{2k}} \cosh \alpha_{2k-1|2k} \\ &\quad - \epsilon_{2k-2} \sqrt{f_{2k-2}} \cosh \alpha_{2k-2|2k-1} \end{aligned} \quad (12)$$

which can be further simplified, using the definition (10), to give

$$\begin{aligned} \tilde{\rho}_{2k-1} &= \epsilon_{2k} \sqrt{f_{2k}} \exp(\pm \alpha_{2k-1|2k}) \\ &\quad - \epsilon_{2k-2} \sqrt{f_{2k-2}} \exp(\mp \alpha_{2k-2|2k-1}). \end{aligned} \quad (13)$$

V. COLLISION AND CONSERVATION LAW

In a small neighbourhood around the collision event, one can consider the change of coordinate systems between two regions in two ways: going from one region to the next anticlockwise or clockwise. The requirement of having the same result in the two cases requires that the composition of the pseudo-Lorentz transformations must give identity after a complete tour around the collision event. This gives the *consistency relation*

$$\prod_{k=1}^N \Lambda(\alpha_{2k-1|2k} - \alpha_{2k-1|2k-2}) = \mathbf{Id}, \quad (14)$$

where we identify the index $i = j + 2N$ with $i = j$. This condition has been obtained recently in a more complicated derivation by Neronov [13] using the existence of common null coordinates. In terms of the Lorentz angles α , this consistency relation is simply the sum rule

$$\sum_{i=1}^{2N} \alpha_{i|i+1} = 0. \quad (15)$$

This relation provides *one constraint*, which can be written in many ways. What we will show is that this relation can be expressed in an extremely intuitive form, which can look either like energy conservation or, equivalently, like momentum conservation.

The main result of this Letter is that, using the junction conditions (13), the sum rule (15) can be written as the conservation law

$$\sum_{k=1}^N \tilde{\rho}_{2k-1} e^{\pm \alpha_{2k-1|j}} = 0, \quad (16)$$

for *any value of the index j* , where we have introduced the generalized relative angle

$$\alpha_{j|j'} = \sum_{i=j}^{j'-1} \alpha_{i|i+1}, \quad (17)$$

if $j < j'$, and $\alpha_{j'|j} = -\alpha_{j|j'}$. To prove Eq. (16) one can simply use Eq. (13) to substitute for $\tilde{\rho}_{2k-1}$ and obtain a sum over exponentials minus another sum which is in fact identical by Eq. (15) and hence they cancel each other out. One must be aware that although one can use Eq. (16) to give many different expressions, there is only one underlying geometrical constraint embodied in (15).

Let us now point out that the conservation law (16) can be written as an *energy conservation law* seen in the j -th reference frame,

$$\sum_{k=1}^N \tilde{\rho}_{2k-1} \gamma_{j|2k-1} = 0, \quad (18)$$

where $\gamma_{j|j'} \equiv \cosh \alpha_{j|j'}$ corresponds to the Lorentz factor between the brane/region j and the brane/region j' and can be obtained, if j and j' are not adjacent, by combining all intermediary Lorentz factors (this is simply using the velocity addition rule of special relativity), or the relative angle formula (17). The index j corresponds to the reference frame with respect to which the conservation rule is written.

But the conservation law (16) can also be written as a *momentum conservation law* in the j -th reference frame,

$$\sum_{k=1}^N \tilde{\rho}_{2k-1} \gamma_{2k-1|j} \beta_{2k-1|j} = 0, \quad (19)$$

with $\gamma_{j|j'} \beta_{j|j'} \equiv \sinh \alpha_{j|j'}$.

Note that the relation (15) implies a strong analogy between the real exponentials (and the hyperbolic cosine and sine), which we are using here, with the complex exponentials (and the usual cosine and sine) by effectively imposing a periodicity.

VI. LIGHT-LIKE BRANES

Our formalism can also be extended to deal with the case of light-like branes [5]. One cannot then introduce a local Lorentz transformation from an adjacent region to the brane frame, as we did above for time-like branes. But we can still consider the coordinate transformation from one of the adjacent regions to the other, which is still of the form (9). Since we now have $\epsilon_{\mathcal{B}} dR = \epsilon_+ f_+ dT_+ = \epsilon_- f_- dT_-$ (with $\epsilon_{\mathcal{B}} = +1$ for a left-moving brane, $\epsilon_{\mathcal{B}} = -1$ for a right-moving brane), one

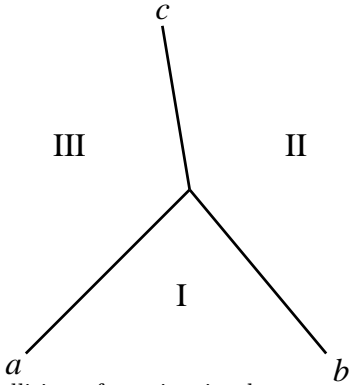


FIG. 2. Collision of two ingoing branes, a and b , to give single outgoing brane c .

finds $\epsilon_+ = \epsilon_-$ (we have been implicitly assuming here that T is a time-like coordinate, i.e. $f > 0$, but it is straightforward to generalize to the case $f < 0$) and

$$e^{\epsilon_{B\alpha}} = \sqrt{f_-/f_+}. \quad (20)$$

For example, in the case of two ingoing light-like branes and two outgoing light-like branes, defining four regions I , II , III and IV , the substitution of the above result in the sum rule (15) immediately yields the DTR (Drayt'Hooft-Redmount) formula [3,4] $f_I f_{III} = f_{II} f_{IV}$. This can be easily generalized to any combination of time-like branes with light-like branes, using the general sum rule for angles (15) and grouping the angles in pairs for the light-like branes.

VII. EXAMPLES

Let us first consider the case of two ingoing branes, a and b , colliding to give a single brane c , separated by the regions I , II and III (see Fig. 2). It is most convenient to express the energy conservation law in the frame of the outgoing brane. One finds $\rho_c = \rho_a \gamma_{a|c} + \rho_b \gamma_{b|c}$. Note that here, the outcome of the collision is completely determined by the situation before collision. Indeed, the outgoing brane velocity and the energy density are interdependent as can be seen in Eq. (8). If one had several outgoing branes then our conservation law would provide only one relation, which should be completed by other information (based for instance on the microphysics of the collision) in order to fully determine the outcome.

A slightly more complicated case, but of direct relevance to the recent ekpyrotic scenario [9] or other works on brane cosmology inspired by the Horava-Witten model [10], is when one of the ingoing branes, a say, is a Z_2 -symmetric orbifold fixed point. We assume that the second brane, b , is not Z_2 -symmetric, otherwise one dimension of spacetime would disappear at the collision. Because of the mirror symmetry about a , one must consider two copies, b and b' , of the incoming brane (see Fig.

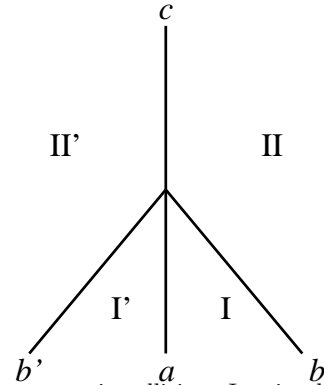


FIG. 3. Z_2 -symmetric collision. Ingoing brane a and outgoing brane c are at orbifold fixed point. Brane b' is mirror image of brane b .

3). We finally assume that the product of the collision is a single Z_2 -symmetric brane, labelled c . Then, Z_2 -symmetry combined with Eq. (15) implies that $\alpha_{a|c} = 0$, i.e. there is no redshift between the ingoing and outgoing Z_2 -symmetric branes. As a consequence, the energy conservation law reads simply $\rho_c = \rho_a + 2\rho_b \gamma_{b|a}$. Note that the total momentum is automatically zero in the frame comoving with a or c , however \dot{R}_c is only zero if ρ_c has the critical value given by Eq. (8). Of course, if one considers the peeling off from an initial Z_2 -symmetric brane, a , of a brane b , then one gets the conservation law $\rho_a = \rho_c + 2\rho_b \gamma_{b|c}$ where the Z_2 -symmetric brane after collision is labelled c . Further applications of our results will be discussed in a separate publication [14].

VIII. CONCLUSIONS

We have presented here a general treatment of the collision of branes in a vacuum spacetime. We have extended to the general case of time-like branes the geometrical constraint characterizing the collision, which was known before only in the special case of light-like branes [3,4]. This constraint can be expressed in the simple form of a sum rule for hyperbolic angles. The general relativistic junction conditions (13) allow us to relate these angles to the energy or momentum of branes at the collision. In this way we obtain extremely simple and intuitive energy and momentum conservation laws, which are analogous to the collision of point particles in two-dimensional special relativity.

One can envisage extending our formalism to the case where the spacetime regions between the branes need be neither empty nor static. One immediate generalization would be to consider a Reissner-Nordstrom-(anti-)de Sitter metric in (1). The formalism would then be unchanged but one would have to supplement the conservation law (15) with the conservation of the brane charges. In general, however, the generalization will be complicated by the need to take into account the junction con-

ditions for the bulk fields and by the possibility that part of the energy at the collision might dissipate in excitations of the bulk field. Nonetheless we believe that our approach, by its simplicity, is likely to be a useful starting point for such a generalization. One application of our formalism will be to shed some light on the evolution of cosmological perturbations through a 3-brane collision [14].

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